MATTER FIELDS IN THE LAGRANGIAN LOOP REPRESENTATION: SCALAR QED

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Abstract

We present the extension of the Lagrangian *loop* gauge invariant representation in such a way to include matter fields. The partition function of lattice compact $U(1)$-Higgs model is expressed as a sum over closed as much as open surfaces. These surfaces correspond to world sheets of loop-like pure electric flux excitations and open electric flux tubes carrying matter fields at their ends. This representation is connected by a duality transformation with the *topological* representation of the partition function (in terms of world sheets of Nielsen-Olesen strings both closed and open connecting pairs of magnetic monopoles). We have simulated numerically the *loop* action equivalent to the Villain form of the action and mapped out the β-γ phase diagram of this model. By virtue of the gauge invariance of this description the equilibrium configurations seems to be reached faster than with the ordinary gauge-variant descriptions.

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1   Introduction

The use of loops for a gauge invariant description of Yang-Mills theories may be traced back to the Mandelstam [1] quantization without potentials. In 1974 Yang [2] noticed the important role of the holonomies for a complete description of gauge theories.

In the early eighties a loop based [3] Hamiltonian approach to quantum electromagnetism was proposed and generalized [4] in 1986 to include the Yang-Mills theory. This Hamiltonian formulation was given in terms of the traces of the holonomies (the Wilson loops) and their temporal loop derivatives as the fundamental objects. They replace the information furnished by the vector potential and the electric field operator, respectively. These gauge invariant operators verify a closed algebra (non canonical) and may be realized on a linear space of loop dependent functions.

Afterwards the loop representation was extended in such a way to include matter fields: the so-called P-representation [5],[6]. This extension includes open paths in addition to the closed ones or loops [1].

The loop approach has many appealing features from the methodological point of view: First, it allows to do away with the first class constraints of the gauge theories (the Gauss law). Second, the formalism only involves gauge invariant objects. This makes it specially well suited to study 'white' objects as mesons and barions in QCD because the wave function will only depend on the paths associated with the physical excitations. Third, all the gauge invariant operators have a simple geometrical meaning when realized in the loop space. Additionally, there is a conceptual motivation: The introduction by Ashtekar [7] of a new set of variables that cast general relativity in the same language as gauge theories allowed to apply loop techniques as a natural non-perturbative description of Einstein's theory. Furthermore, the loop representation appeared as the most appealing application of the loop techniques to this problem [8], [9]. Thus, it was realized that this loop formalism goes beyond a simple gauge invariant description and in fact it provides a natural geometrical framework to treat gauge theories and quantum gravity.

The non-canonical nature of the loop algebra have made elusive a Lagrangian loop formalism counterpart of the original Hamiltonian one [10], [11]. In a previous paper [12] we showed a natural procedure in order to cast loops into the Lagrangian formalism. In reference [12], it was considered the lattice 4D pure QED model and

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[1] Here we will use the term 'loop' for the configurations in presence of matter fields albeit in a relaxed sense which covers both closed as much as open paths.

it was written the action for the loops as a sum over their closed world sheets.

In the present paper we shall continue with the program of setting up the Lagrangian correlative of the Hamiltonian loop formalism. Concretely, we extend this Lagrangian loop approach in such a way to include matter fields.

We consider the lattice compact version of the scalar electrodynamics $SQED_c$ which describes the interaction of a compact gauge field θ_μ with the scalar field $\phi = |\phi|e^{i\varphi}$. The self-interaction of the scalar field is given by the potential $\lambda(|\phi|^2 - \phi_0^2)^2$. For simplicity we shall consider the limit $\lambda \to \infty$ which freezes the radial degree of freedom of the Higgs field (it is known that the numerical results obtained already at $\lambda = 1$ are indistinguishable from the frozen case). Thus the dynamical variable is compact, i.e. $\varphi \in (-\pi, \pi]$. This model is known to possess three phases, namely confining, Higgs and Coulomb [13]. The Higgs phase splits into a region where magnetic flux can penetrate in form of vortices (Nielsen-Olesen strings) and a region where the magnetic flux is completely expelled [14], the relativistic version of Meissner effect in superconductivity. Relying on this, we call this two subregions: Higgs I and II in analogy with superconducting materials.

2 The Loop Representation: Hamiltonian and Lagrangian Formulations

The Hamiltonian loop representation of the scalar compact QED in a lattice – with sites denoted by x, links denoted by l and plaquettes by p – is given in terms of the fundamental gauge operators of the P-representation, the $\hat{\Phi}(P_x^y)$ defined by [2]

$$\hat{\Phi}(P_x^y) = \hat{\phi}^\dagger(x)\hat{U}(P_x^y)\hat{\phi}(y) = \hat{\phi}^\dagger(x) \prod_{l \in P} \hat{U}(l)\hat{\phi}(y) \tag{1}$$

($\hat{U}(l)$ are the lattice gauge group operators, $\hat{\phi}(x)$ are the matter field operators) and their conjugate momenta: the electric field operator $\hat{E}(l)$ and the $\hat{\Pi}^\dagger(x)$ and $\hat{\Pi}(x)$. They obey the commutation relations

$$\left[\hat{E}(l), \hat{\Phi}(P_x^y)\right] = N_l(P)\hat{\Phi}(P_x^y)$$

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[2] P_x^y comprises open as much as closed paths or loops C at fixed t, representing "electromesons" and pure gauge excitations respectively. In this last case $\hat{\Phi}(C) \equiv \hat{W}(C)$ i.e. it reduces to the Wilson loop operator.

$$\left[\hat{\Pi}^\dagger(x), \hat{\Phi}(P_y^z)\right] = -i\delta_{xy}\hat{\Phi}(P_y^z)$$

$$\left[\hat{\Pi}(x), \hat{\Phi}(P_y^z)\right] = -i\delta_{xz}\hat{\Phi}(P_y^z) \tag{2}$$

where $N_l(P)$ is the number of times the link l appears in the path P.
The $\hat{\Phi}(P_x^y)$ operators are the creation operators of the loops i.e.

$$\hat{\Phi}(P_x^y)|0> = \hat{\phi}^\dagger(x)\hat{U}(P_x^y)\hat{\phi}(y)|0> = |P_x^y> \tag{3}$$

where $|0>$ is the zero loop state (strong coupling vacuum of the system).

In order to cast the preceding loop description in the Lagrangian formalism let us consider the partition function for the Villain form of the lattice action[3] which is given by

$$Z = \int(d\theta)\sum_n\int(d\varphi)\sum_k\exp(-\frac{\beta}{2}\parallel\nabla\theta - 2\pi n\parallel^2 -\frac{\kappa}{2}\parallel\nabla\varphi - 2\pi k - \theta\parallel^2), \tag{4}$$

where we use the notations of the calculus of differential forms on the lattice of [15]. In the above expression: $\beta = \frac{1}{e^2}$, θ is a real compact 1-form defined in each link of the lattice and φ is a real compact 0-form defined on the sites of the lattice, ∇ is the co-boundary operator, n are integer 2-forms defined at the lattice plaquettes, and k integer 1-forms, and $\parallel . \parallel^2 = <.,.>$.

If we use the Poisson summation formula $\sum_n f(n) = \sum_s\int_{-\infty}^{\infty}d\phi f(\phi)e^{2\pi i\phi s}$ for each of the integer variables, the partition function (4) transforms into

$$Z = \sum_s\sum_t\int(d\theta)\int(d\varphi)\int_{-\infty}^{+\infty}(d\psi)\int_{-\infty}^{+\infty}(d\chi)\exp(-\frac{\beta}{2}\parallel\nabla\theta - 2\pi\psi\parallel^2$$
$$-\frac{\kappa}{2}\parallel\nabla\varphi - 2\pi\chi - \theta\parallel^2)e^{i2\pi<s,\psi>}e^{i2\pi<t,\chi>}. \tag{5}$$

Integrating in the ψ and χ variables

$$Z \propto \sum_s\sum_t\int(d\theta)\int(d\varphi)\exp(-\frac{1}{2\beta}\parallel s\parallel^2 -\frac{1}{2\kappa}\parallel t\parallel^2)$$
$$\times e^{i<s,\nabla\theta>}e^{i<t,\nabla\varphi-\theta>}. \tag{6}$$

Using the partial integration rule $<\psi,\nabla\phi> = <\partial\psi,\phi>$ ($\partial = *\nabla*$ is the boundary operator which maps k-forms into (k-1)-forms and where $*$ is the duality operation

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[3] The choice of the Villain form instead of the ordinary Wilson form is only done for simplicity, with the Wilson action it is also possible to repeat everything of what follows.

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which maps k-forms into (4-k)-forms) and integrating over the compact φ and θ we get the constrains $\delta(\partial t = 0)$ and $\delta(\partial s = t)$ and thus, we finally arrive to

$$Z \propto \sum_s \exp(-\frac{1}{2\beta} \parallel s \parallel^2 -\frac{1}{2\kappa} \parallel \partial s \parallel^2) \tag{7}$$

or

$$Z \propto \sum_s \exp(-\frac{1}{2\beta} < s, \frac{\nabla \partial + M^2}{M^2} s >), \tag{8}$$

where $M^2 = \frac{\kappa}{\beta}$ is the mass acquired by the gauge field due to the Higgs mechanism. If we consider the intersection of one of the surfaces defined by the integer 2-forms s (open and closed surfaces) with a $t = constant$ plane we get spatial paths or 'loops'. It is easy to show that the creation operator of those paths is just the the $\hat{\Phi}(P_x^y)$ operator. Repeating the steps from Eq.(4) to Eq.(7) we get for $< \hat{\Phi}(P_x^y) >$

$$< \hat{\Phi}(P_x^y) > \propto \sum_s \exp(-\frac{1}{2\beta} \parallel s \parallel^2 -\frac{1}{2\kappa} \parallel \partial s + P_x^y \parallel^2). \tag{9}$$

Thus, we have arrived to an expression of the partition function in terms of the world sheets of electric string-like configurations: the *loop* (Lagrangian) representation. In this representation the matter fields are naturally introduced by means of open surfaces which are the world sheets of open paths representing the 'meson-like' configurations. In other words, as it is shown in Fig.1, cutting the above open surfaces with planes $t = constant$ we get the corresponding quantum Hamiltonian description in terms of open paths P_x^y (the P-representation).

3 Duality and the topological representation

Another equivalent description of the Villain form is the *topological* representation in terms of the topological objects. As our model has two compact variables we have two topological excitations: monopoles and Nielsen-Olesen strings [16]. The BKT expression for the partition function of compact scalar QED is obtained via the Banks-Kogut-Myerson transformation [17] (see Appendix) and is given by

$$Z \propto \sum_{n(m)} \exp(-2\pi^2 \beta < *n(m), \frac{M^2}{\partial \nabla + M^2} * n(m) >) \tag{10}$$

where $m = \partial * n$ are closed integer 1-forms attached to links which represent monopole loops and $*n(m) = *n - \partial * q$ are integer 2-forms attached to plaquettes corresponding to the world sheets of both Dirac and Nielsen-Olesen strings (with monopole loops as borders). Thus, comparing (8) and (10) we can observe a complete parallelism: in both representations we have a sum over surfaces, and intersecting with a plane $t = constant$ we get closed as much as open strings with point charges at their ends. In the first case this string-like excitations are 'electric' whilst in the second they are 'magnetic'. Furthermore, there is a duality transformation connecting the confining and Higgs II sectors of the phase diagram. We want to remark that there is a slight difference between both equivalent descriptions. In the BKT representation monopoles occur at the ends of both the Nielsen-Olesen strings (physical excitations) and the Dirac strings (non physical gauge-variant objects) so we have the corresponding two types of world sheets mixed in the 2-form $*n(m)$ of equation (10). On the other hand the gauge invariant *loop* description is simpler and completely transparent from the geometrical point of view.

4   Numerical Analysis

We have performed a standard Metropolis Monte Carlo simulation with the loop action of (7). We have worked on a hypercubical lattice. Basic variables are integers $n_{\mu\nu}(r)$ attached to plaquettes $p = (r; \mu, \nu)$, where r denotes the site and $\mu, \nu = 1 - 4$ two directions. These variables describe the world-sheets of the loop-excitations.

The loop action of (7) is equivalent to the Villain form. There were not numerical results of the phase diagram (as far as we know) for the Villain action of SQED_c. Thus, in order to map out the phase diagram, we made preliminary Monte Carlo runs along lines of fixed β or γ on a 6^4 lattice. Typically, 5,000 iterations were done at each coupling and measurements of the specific heat were taken. Then the coupling was changed by ± 0.005, and another 5,000 iterations were made, etc. The resulting crude phase diagram (three lines: confining-Coulomb, confining-Higgs and Higgs-Coulomb) proved very helpful in guiding our larger scale simulations.

The next step was to localize accuratelly the transition lines in a 8^4 lattice. With this aim, we have performed runs of 40,000 iterations after 15,000 termalization sweeps per-point on the plane $\beta - \gamma$. The phase diagram is shown in Fig. 2.

It is worth while to mention that the *loop* description avoids the problem of summing over gauge redundant configurations. This is reflected in the fact that when simulating the *loop* action the convergence to the equilibrium configurations seems to be quite faster than when using the ordinary action in terms of the fields. Fur-

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thermore, we observed the absence of the strong metastability previously observed in Monte Carlo analysis of the SQED$_c$ lattice theory with the Wilson action, which makes very difficult the numerical analysis of the phase transition critical exponents. This fact is one of the main advantages for using the action defined in eq. (7).

5   Conclusions

First, we have shown how to introduce matter fields in the Lagrangian *loop* representation. It turns out that the partition function of gauge-Higgs theory can be represented as the sum over world sheets of loops (open and closed).

Second, a correspondence between the *loop* and the BKT descriptions is patent and suggests a duality transformation connecting the confining and the Higgs II regions of the phase diagram. Perhaps this connection found in the lattice also holds in the continuum, supporting the claims on the existence of a new phase in QED [18].

Third, simulating numerically a loop-action equivalent to the Villain form of the SQED model (with frozen radial degree of freedom) we mapped out its phase diagram. It turns out that it is very similar to the corresponding phase diagram for the Wilson form [19], [20]. Additionally, these results confirm those previously obtained for the same model but using the Hamiltonian *loop* approach [21], because the mapped phase diagram is qualitatively equal in both cases.

The next step will be the implementation of the Pauli exclusion principle in the Lagrangian loop approach in order to include fermionic fields. This task has been accomplished in the context of the Hamiltonian *loop* formalism in reference [5] where a transparent geometrical description of 'full' QED was given.

Appendix

To obtain the monopole representation (10) we start with the Villain form (4) and fix the gauge $\varphi = 0$. Then, we parameterize the $n = \nabla q + \bar{n}(v)$, where q run over arbitrary 1-forms and v over all co-closed 3-forms ($\nabla v = 0$). $\bar{n}(v)$ is a solution of $\nabla n = v$. If we perform a translation $k \to k - q$ we get the expression

$$Z \propto \int_{-\infty}^{+\infty} (dA) \sum_{\substack{v \\ (\nabla v = 0)}} \sum_{k} \exp(-\frac{\beta}{2} \| \nabla A - 2\pi\bar{n} \|^2 - \frac{\kappa}{2} \| A + 2\pi k \|^2) \quad (11)$$

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where $A = \theta - 2\pi q$ is a non-compact variable. By shifting A by $2\pi k$ we find dependence on the combination $\bar{n} + \nabla k$, which turns to be a solution of $\nabla n = v$ so we can eliminate the k variable

$$Z \propto \int_{-\infty}^{+\infty} (dA) \sum_{\substack{v \\ (\nabla v = 0)}} \exp(-\frac{\beta}{2} \parallel \nabla A - 2\pi\bar{n} \parallel^2 -\frac{\kappa}{2} \parallel A \parallel^2) \qquad (12)$$

and performing the gaussian integration we obtain

$$Z \propto \sum_{\substack{v \\ (\nabla v = 0)}} \exp(-2\pi^2\beta < \bar{n}(v), \frac{M^2}{\nabla\partial + M^2}\bar{n}(v) >) \qquad (13)$$

Performing a duality transformation we get (10) where $m = *v$ are now integer closed 1-forms ($\partial m = 0$) and $*n(m) = *\bar{n}(v) + \partial * q$ so $\partial * n = m$. It is possible to express (13) in terms of the k instead of the n variables which reflects the presence of Dirac and Nielsen-Olesen sheets.

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Figure captions

- Figure 1: Scalar QED surfaces as string propagation.

- Figure 2: Phase diagram of scalar QED.

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